Math 43 Midterm 3 Review Solutions

[1] [a] $a_3 = a_1 + 2d = x - 2y$ and $a_8 = a_1 + 7d = 4y - 3x$

$a_1 + 7d - (a_1 + 2d) = 4y - 3x - (x - 2y)$

$d = \frac{6}{5}y - \frac{4}{5}x$

$a_1 = x - 2y - 2(\frac{6}{5}y - \frac{4}{5}x) = \frac{13}{5}x - \frac{22}{5}y$

$a_{11} = a_1 + 10d = \boxed{\frac{38}{5}y - \frac{27}{5}x}$

[b] $a_5 = a_1 r^4 = \dfrac{x^8}{2y^5}$ and $a_8 = a_1 r^7 = 4x^3 y$

$$\frac{a_1 r^7}{a_1 r^4} = \frac{4x^3 y}{\frac{x^8}{2y^5}}$$

$r = 2x^{-\frac{5}{3}}y^2$

$$a_1 = \frac{4x^3 y}{(2x^{-\frac{5}{3}}y^2)^7} = \frac{x^{\frac{44}{3}}}{32y^{13}}$$

$$a_{13} = a_1 r^{12} = \boxed{\frac{128y^{11}}{x^{\frac{16}{3}}}}$$

[2] $-73 + 7(n-1) = 529 \quad \rightarrow \quad 7(n-1) = 602 \quad \rightarrow \quad n - 1 = 86 \quad \rightarrow \quad n = 87$

$S_{87} = \dfrac{87}{2}(-73 + 529) = \boxed{19{,}836}$

[3] $(-1)^3 3(3-4) \;+\; (-1)^4 4(4-4) \;+\; (-1)^5 5(5-4) \;+\; (-1)^6 6(6-4) \;+\; (-1)^7 7(7-4) \;+\; (-1)^8 8(8-4)$

$= 3 + 0 - 5 + 12 - 21 + 32$

$= \boxed{21}$

[4] [a] PROOF:

Basis step: $1^3 = 1 = \dfrac{1^2(1+1)^2}{4}$

Inductive step: Assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$ for some particular but arbitrary integer $k \ge 1$

Prove $1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \dfrac{(k+1)^2(k+2)^2}{4}$

$$\begin{aligned}
&1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 \\
&= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{(k+1)^2}{4}(k^2 + 4(k+1)) \\
&= \frac{(k+1)^2}{4}(k^2 + 4k + 4) \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

So, by mathematical induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all integers $n \ge 1$

[b] PROOF:

Basis step: $\sum_{i=0}^{0}(2i+1)3^{i-1} = 1\cdot 3^{-1} = \frac{1}{3} = \frac{1+0\cdot 3^{1}}{3}$

Inductive step: Assume $\sum_{i=0}^{k}(2i+1)3^{i-1} = \frac{1+k3^{k+1}}{3}$ for some particular but arbitrary integer $k \geq 0$

Prove $\sum_{i=0}^{k+1}(2i+1)3^{i-1} = \frac{1+(k+1)3^{k+2}}{3}$

$$\sum_{i=0}^{k+1}(2i+1)3^{i-1}$$
$$= \sum_{i=0}^{k}(2i+1)3^{i-1} + (2(k+1)+1)3^{(k+1)-1}$$
$$= \frac{1+k3^{k+1}}{3} + (2k+3)3^{k}$$
$$= \frac{1+k3^{k+1}+3(2k+3)3^{k}}{3}$$
$$= \frac{1+k3^{k+1}+(2k+3)3^{k+1}}{3}$$
$$= \frac{1+(k+2k+3)3^{k+1}}{3}$$
$$= \frac{1+(3k+3)3^{k+1}}{3}$$
$$= \frac{1+3(k+1)3^{k+1}}{3}$$
$$= \frac{1+(k+1)3^{k+2}}{3}$$

So, by mathematical induction, $\sum_{i=0}^{n}(2i+1)3^{i-1} = \frac{1+n3^{n+1}}{3}$ for all integers $n \geq 0$

[c] PROOF:

Basis step: $a + ar = a(1+r) = \frac{a(r^{2}-1)}{r-1}$

Inductive step: Assume $a + ar + ar^{2} + \cdots + ar^{k} = \frac{a(r^{k+1}-1)}{r-1}$ for some particular but arbitrary integer $k \geq 1$

Prove $a + ar + ar^{2} + \cdots + ar^{k+1} = \frac{a(r^{k+2}-1)}{r-1}$

$$a + ar + ar^{2} + \cdots + ar^{k+1}$$
$$= a + ar + ar^{2} + \cdots + ar^{k} + ar^{k+1}$$
$$= \frac{a(r^{k+1}-1)}{r-1} + ar^{k+1}$$
$$= \frac{a}{r-1}[(r^{k+1}-1) + r^{k+1}(r-1)]$$
$$= \frac{a}{r-1}(r^{k+1} - 1 + r^{k+2} - r^{k+1})$$
$$= \frac{a(r^{k+2}-1)}{r-1}$$

So, by mathematical induction, $a + ar + ar^{2} + \cdots + ar^{n} = \frac{a(r^{n+1}-1)}{r-1}$ for all integers $n \geq 1$

[d] PROOF:

Basis step: $\sum_{i=1}^{1} \frac{3}{(i+3)(i+2)} = \frac{3}{(4)(3)} = \frac{1}{4} = \frac{1}{1+3}$

Inductive step: Assume $\sum_{i=1}^{k} \frac{3}{(i+3)(i+2)} = \frac{k}{k+3}$ for some particular but arbitrary integer $k \geq 1$

Prove $\sum_{i=1}^{k+1} \frac{3}{(i+3)(i+2)} = \frac{k+1}{k+4}$

$$\sum_{i=1}^{k+1} \frac{3}{(i+3)(i+2)}$$

$$= \sum_{i=1}^{k} \frac{3}{(i+3)(i+2)} + \frac{3}{((k+1)+3)((k+1)+2)}$$

$$= \frac{k}{k+3} + \frac{3}{(k+4)(k+3)}$$

$$= \frac{k(k+4)+3}{(k+4)(k+3)}$$

$$= \frac{k^2+4k+3}{(k+4)(k+3)}$$

$$= \frac{(k+1)(k+3)}{(k+4)(k+3)}$$

$$= \frac{k+1}{k+4}$$

So, by mathematical induction, $\sum_{i=1}^{n} \frac{3}{(i+3)(i+2)} = \frac{n}{n+3}$ for all integers $n \geq 1$

[5] $0.4 + 0.072 + 0.00072 + 0.0000072 + \cdots$

$$= \frac{4}{10} + \left(\frac{72}{1000} + \frac{72}{100000} + \frac{72}{10000000} + \cdots \right)$$

$$= \frac{2}{5} + \left(\frac{\frac{72}{1000}}{1 - \frac{1}{100}} \right)$$

$$= \frac{2}{5} + \left(\frac{\frac{72}{1000}}{\frac{99}{100}} \right)$$

$$= \frac{2}{5} + \frac{72}{1000}\frac{100}{99}$$

$$= \frac{2}{5} + \frac{4}{55}$$

$$= \boxed{\frac{26}{55}}$$

[6] $\frac{200!}{4! \cdot 196!} = \frac{200 \cdot 199 \cdot 198 \cdot 197 \cdot 196!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 196!} = \boxed{64{,}684{,}950}$

[7] NOTE: The first factors in the denominator form an arithmetic sequence, and the second factors form a geometric sequence.

$$\sum_{n=1}^{9} \frac{1}{(7-3(n-1))\cdot 3(2)^{n-1}} = \boxed{\sum_{n=1}^{9} \frac{1}{3(10-3n)(2)^{n-1}}}$$

NOTE: To find the upper limit of summation, either solve

$$7-3(n-1)=-17 \quad \text{or} \quad 3(2)^{n-1}=768$$
$$-3(n-1)=-24 \qquad 2^{n-1}=256$$
$$n-1=8 \qquad n-1=8$$
$$n=9 \qquad n=9$$

[8] The general term is $\binom{11}{r}(2x^5)^{11-r}(-3x^2)^r = \binom{11}{r}2^{11-r}(-3)^r(x^5)^{11-r}(x^2)^r = \binom{11}{r}2^{11-r}(-3)^r x^{55-3r}$

$$55-3r=34 \quad \rightarrow \quad r=7 \quad \rightarrow \quad \binom{11}{7}2^{11-7}(-3)^7 = \boxed{-11{,}547{,}360}$$

[9] $4(0.97)^{2(3)-1} + 4(0.97)^{2(4)-1} + 4(0.97)^{2(5)-1} + \cdots$

$= 4(0.97)^5 + 4(0.97)^7 + 4(0.97)^9 + \cdots$

$= \dfrac{4(0.97)^5}{1-(0.97)^2}$

$\approx \boxed{58.1207}$

[10]
$$a_2 = 2a_1-3 = 2(4)-3 = 5$$
$$a_3 = 2a_2-3 = 2(5)-3 = 7$$
$$a_4 = 2a_3-3 = 2(7)-3 = 11$$
$$a_5 = 2a_4-3 = 2(11)-3 = 19$$

$\boxed{4, 5, 7, 11, 19}$

The sequence is neither arithmetic nor geometric. The differences are $1, 2, 4, 8$ which are not constant.

The ratios are $\frac{5}{4}, \frac{7}{5}, \frac{11}{7}, \frac{19}{11}$ which are also not constant.

[11] [a] $1(3x)^6(-2y)^0 + 6(3x)^5(-2y)^1 + 15(3x)^4(-2y)^2 + 20(3x)^3(-2y)^3$

$+ 15(3x)^2(-2y)^4 + 6(3x)^1(-2y)^5 + 1(3x)^0(-2y)^6$

$= \boxed{729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6}$

[b] $1(\sqrt{x})^4\left(-\frac{2}{x}\right)^0 + 4(\sqrt{x})^3\left(-\frac{2}{x}\right)^1 + 6(\sqrt{x})^2\left(-\frac{2}{x}\right)^2 + 4(\sqrt{x})^1\left(-\frac{2}{x}\right)^3 + 1(\sqrt{x})^0\left(-\frac{2}{x}\right)^4$

$= x^2 + 4x^{\frac{3}{2}}(-2x^{-1}) + 6x(4x^{-2}) + 4x^{\frac{1}{2}}(-8x^{-3}) + 16x^{-4}$

$= \boxed{x^2 - 8x^{\frac{1}{2}} + 24x^{-1} - 32x^{-\frac{5}{2}} + 16x^{-4}}$

[12] $800(0.9)^{n-1} = 3.34 \rightarrow (0.9)^{n-1} = 0.004175 \rightarrow \ln(0.9)^{n-1} = \ln 0.004175 \rightarrow$

$(n-1)\ln 0.9 = \ln 0.004175 \rightarrow n-1 = \dfrac{\ln 0.004175}{\ln 0.9} \rightarrow n = 1 + \dfrac{\ln 0.004175}{\ln 0.9} \approx 53$

Since year 1 corresponded to 1998, year 2 corresponded to 1999, year 3 corresponded to 2000,

EJ's car was sold for scrap in $1998-1+53=2050$

[13] CJ's total rent will be $\frac{24}{2}(2\times 400+(24-1)(7)) = \$11{,}532$.

DJ's total rent will be $\dfrac{380(1.02^{24}-1)}{1.02-1} = \$11{,}560.31$. So, DJ will have paid $\$28.31$ more rent.